An Efficient Dynamic hp-Discontinuous Galerkin Formulation for Time-Domain Electromagnetics

ESCO 2012, Pilsen, Czech Republic

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Outline

- Semi-Discrete Formulation of the Discontinuous Galerkin (DG) Method for Maxwell’s Equations
  - Weak formulation
  - A priori error estimation
- The DG Method on Non-Regular Meshes
  - Flux computation
  - Efficiency Tweaks
- Adaptive Mesh Refinement
- Examples
We consider Maxwell’s equations:

\[ \nabla \times \mathbf{H} = \frac{\partial}{\partial t} \mathbf{D} + \mathbf{J} \quad \text{Ampere’s law} \]

\[ \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B} \quad \text{Faraday’s law} \]

\[ \nabla \cdot \mathbf{D} = \rho \quad \text{Gauss’ law} \]

\[ \nabla \cdot \mathbf{B} = 0 \quad \text{Gauss’ law of magnetic fields} \]

### Constitutive equations

- \( \mathbf{B} = \mu \mathbf{H} \)
- \( \mathbf{D} = \epsilon \mathbf{E} \)
- \( \mathbf{J}_\kappa = \kappa \mathbf{E} \)

### Units

- Electric field: \( \mathbf{E} \) = V/m = Volt / meter
- Electric flux density: \( \mathbf{D} \) = As/m² = Ampère second / meter²
- Magnetic field: \( \mathbf{H} \) = A/m = Ampère / meter
- Magnetic flux density: \( \mathbf{B} \) = Vs/m² = Volt second / meter²
- Current density: \( \mathbf{J} \) = A/m² = Ampère / meter²
**DG for Maxwell -- Semi-discrete Formulation**

Weak semi-discrete DG formulation of Maxwell’s equations in three-dimensional space:

Find $e_P^i$ and $h_P^i$ such that $\forall j = 1..N$, $\forall q = 1..P$

\[
\sum_{i,p} \delta_{ij} \left( \int_{\Omega_j} d^3 r \mu \varphi_i^p \varphi_j^q \right) d_t h_i^p + \int_{\partial \Omega_j} d^2 r (n \times \tilde{E}^*) \varphi_j^q - \sum_{i,p} \delta_{ij} \left( \int_{\Omega_j} d^3 r \varphi_i^p (\nabla \varphi_j^q) \right) \times e_i^p = 0
\]

\[
\sum_{i,p} \delta_{ij} \left( \int_{\Omega_j} d^3 r \varepsilon \varphi_i^p \varphi_j^q \right) d_t e_i^p - \int_{\partial \Omega_j} d^2 r (\tilde{n} \times \tilde{H}^*) \varphi_j^q + \sum_{i,p} \delta_{ij} \left( \int_{\Omega_j} d^3 r \varphi_i^p (\nabla \varphi_j^q) \right) \times h_i^p = 0
\]

**Mass term**  **Flux term**  **Rigidity term**

with basis functions $\varphi_i^p$ and test functions $\varphi_j^q \in H^1$
DG for Maxwell -- Semi-discrete Formulation

A priori error estimation

- We consider the representation of two sets of initial conditions in the FE space
  1. Normal distribution \((C^\infty)\)
  2. Trapezoidal distribution \((C^0)\)

- Why is there such a BIG difference?
  - Smoothness prerequisite violated
  - Considering the slope...
A priori error estimation

- We consider the representation of two sets of initial conditions in the FE space
  1. Normal distribution \((C^\infty)\)
  2. Trapezoidal distribution \((C^0)\)

- Consider the mesh dependent norm [Bey, Oden (1996) (simplified form)]

\[
\|E\|_{hp} \leq C_1 \left\{ \sum_i \left[ C_2 \frac{(\Delta x)^{2\nu - 1}}{p_i^{2s-2}} \|\tilde{U}\|_2 \right] \right\}^{1/2}
\]

where \(\nu = \min(p_i + 1, s)\)

and \(s\) is the regularity index
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DG for Maxwell -- Non-regular meshes

High level hanging nodes using precomputed fluxes

- Hanging nodes can easily be included in the DG framework
DG for Maxwell -- Non-regular meshes

High level hanging nodes using precomputed fluxes

- Hanging nodes can easily be included in the DG framework

Elements of h-refinement level $H$

Elements of h-refinement level $H-1$
DG for Maxwell -- Non-regular meshes

High level hanging nodes using precomputed fluxes

- Flux from right to left

Elements of h-refinement level H

Elements of h-refinement level H-1
High level hanging nodes using precomputed fluxes

- Split into four (generally N) partial fluxes according to element overlap
DG for Maxwell -- Non-regular meshes

High level hanging nodes using precomputed fluxes

- Flux from left to right
High level hanging nodes using precomputed fluxes

- Combine four (generally \( N \)) partial fluxes into one according to element overlap
DG for Maxwell -- Non-regular meshes

High level hanging nodes using precomputed fluxes

- Combine four (generally N) partial fluxes into one according to element overlap
DG for Maxwell -- Non-regular meshes

- **Flux term** (for box elements)

\[
\int_{\partial \Omega_j} d^2 r (\mathbf{n} \times \mathbf{\tilde{E}}^*) \varphi_j^q = \left( \int_{\partial \Omega_j} (n_y \mathbf{\tilde{E}}_z^* - n_z \mathbf{\tilde{E}}_y^*) \varphi_j^q d^2 r \right) + \left( \int_{\partial \Omega_j} (n_z \mathbf{\tilde{E}}_x^* - n_x \mathbf{\tilde{E}}_z^*) \varphi_j^q d^2 r \right) + \left( \int_{\partial \Omega_j} (n_x \mathbf{\tilde{E}}_y^* - n_y \mathbf{\tilde{E}}_x^*) \varphi_j^q d^2 r \right)
\]

\[
\int_{\partial \Omega_{j,y}^+} \mathbf{\tilde{E}}_z^* \varphi_j^q dxdz \quad \Rightarrow \quad \int_{\partial \Omega_{j,y}^+} \left( \sum_{i,p} e_{i,z}^p \varphi_i^p \right)^* \varphi_j^q dxdz
\]

\[
= \frac{1}{2} \sum_p \int_{\partial \Omega_{j,y}^+} \left( e_{j,x}^p \varphi_j^p \varphi_{j,x}^p + e_{j,y}^p \varphi_j^p \varphi_{j,y}^p \right) \varphi_j^q dxdz
\]

\[
= \frac{1}{2} \sum_p \int_{\partial \Omega_{j,y}^+} \left( e_{j,x}^p \varphi_j^p(x) \varphi_{j,x}^p(1) \varphi_j^p(z) + e_{j,y}^p \varphi_j^p(x) \varphi_{j,y}^p(-1) \varphi_j^p(z) \right) \varphi_j^q dxdz
\]
DG for Maxwell -- Non-regular meshes

- **Flux term**

\[
\frac{1}{2} \sum_p \int_{\partial \Omega^{j,y}_{+}} \left( e_j^p \varphi_j^{px}(x) \varphi_j^{py}(1) \varphi_j^{pz}(z) + e_{\text{ngbr}}^p \varphi_{\text{ngbr}}^{px}(x) \varphi_{\text{ngbr}}^{py}(-1) \varphi_{\text{ngbr}}^{pz}(z) \right) \varphi_j^q \, dx \, dz
\]

- Split in integral terms of boundary values (1, -1)
- and integrals of the form

1. \[
\int_{x_j} \varphi_j^{px}(x) \varphi_j^{qx}(x) \, dx
\]

2. \[
\int_{x_j \cap x_{\text{ngbr}}} \varphi_{\text{ngbr}}^{px}(x) \varphi_j^{qx}(x) \, dx
\]

**Complete** overlap of basis and test function (same element)

**Partial** overlap of basis and test function (neighboring element)
DG for Maxwell -- Non-regular meshes

- **Flux term**

\[
\frac{1}{2} \sum_p \int_{\partial \Omega^+_{j,y}} \left( e_j^p \varphi_j^{px} (x) \varphi_j^{py} (z) + e_{ngbr} \varphi_{ngbr}^{px} (x) \varphi_{ngbr}^{py} (-1) \varphi_{ngbr}^{pz} (z) \right) \varphi_j^q \, dx \, dz
\]

*Partial overlap:*

2. \[ \int_{x_j \cap x_{ngbr}} \varphi_{ngbr}^{px} (x) \varphi_j^{qx} (x) \, dx \]

**Partial overlap of basis and test function (neighboring element)**

\[ \Delta H = 1 \]

\[
\int_{x_{j0} + \frac{1}{\Delta H+1} \Delta x}^{x_{j0} + \frac{2}{\Delta H+1} \Delta x} \varphi_j^{px} (x) \varphi_j^{qx} (x) \, dx
\]

\[
\int_{x_{j0} + \frac{0}{\Delta H+1} \Delta x}^{x_{j0} + \frac{1}{\Delta H+1} \Delta x} \varphi_j^{px} (x) \varphi_j^{qx} (x) \, dx
\]
DG for Maxwell -- Non-regular meshes

- **Flux term**

\[
\frac{1}{2} \sum_p \int_{\partial \Omega_{j,y}^+} \left( e_j^p \varphi_j^p (x) \varphi_j^p y (1) \varphi_j^p z (z) + e_{\text{ngbr}}^p \varphi_{\text{ngbr}}^p (x) \varphi_{\text{ngbr}}^p y (-1) \varphi_{\text{ngbr}}^p z (z) \right) \varphi_{j}^q dxdz
\]

- **Partial overlap:**

\[
2. \int_{x_j \cap x_{\text{ngbr}}} \varphi_{\text{ngbr}}^p (x) \varphi_j^q x (x) dx
\]

\[\Delta H = 2\]

\[
\int_{x_{j0} + \frac{1}{\Delta H+1} \Delta x}^{x_{j0} + \frac{2}{\Delta H+1} \Delta x} \varphi_j^p x (x) \varphi_{j}^q x (x) dx
\]

\[
\int_{x_{j0} + \frac{2}{\Delta H+1} \Delta x}^{x_{j0} + \frac{3}{\Delta H+1} \Delta x} \varphi_j^p x (x) \varphi_{j}^q x (x) dx
\]

\[
\int_{x_{j0} + \frac{3}{\Delta H+1} \Delta x}^{x_{j0} + \frac{4}{\Delta H+1} \Delta x} \varphi_j^p x (x) \varphi_{j}^q x (x) dx
\]
DG for Maxwell -- Non-regular meshes

- **Flux term**

\[
\frac{1}{2} \sum_p \int_{\partial \Omega^+_{j,y}} \left( e^p_j \varphi^p_j(x) \varphi^p_y(1) \varphi^p_z(z) + e^p_{\text{ngbr}} \varphi^p_x(\text{ngbr}) \varphi^p_y(\text{ngbr})(-1) \varphi^p_z(\text{ngbr}) \right) \varphi^q_j \, dx \, dz
\]

- **Partial overlap:**

2. \[
\int_{x_j \cap x_{\text{ngbr}}} \varphi^p_{\text{ngbr}}(x) \varphi^q_j(x) \, dx
\]

---

$\Delta H = 3$

\[
\int_{x_j + \frac{1}{2} \Delta x}^{x_j + \frac{1}{2} \Delta x} \varphi^p_j(x) \varphi^q_j(x) \, dx
\]

Partial overlap of basis and test function (neighboring element)
DG for Maxwell -- Non-regular meshes

- **Flux term**

  - Precompute (analytically) and tabulate
    \[
    \int_{x_{j0} + \frac{h}{\Delta H+1} \Delta x}^{x_{j0} + \frac{h+1}{\Delta H+1} \Delta x} \varphi^p_j(x) \varphi^q_j(x) \, dx
    \]
    
    - for all \( \Delta H \) (< 7 => \((1/2)^6)^2 = 4096\) neighboring elements) and
    - for all combinations of partial overlaps for the respective \( \Delta H \)
    - all combinations of basis and test functions \( (p_x,q_x) \)
  
  - 3-dimensional matrices of size \( \Delta H \times P \times P \)
DG for Maxwell -- Non-regular meshes

**Flux term**

- Precompute (analytically) and tabulate
  \[ \int_{x_j_0 + \frac{h}{\Delta H + 1} \Delta x}^{x_j_0 + \frac{h + 1}{\Delta H + 1} \Delta x} \varphi_{j}^{p_x}(x) \varphi_{j}^{q_x}(x) \, dx \]
  - for all \( \Delta H (< 7 \Rightarrow (1/2^6)^2 = 4096 \) neighboring elements) and
  - for all combinations of partial overlaps for the respective \( \Delta H \)
  - all combinations of basis and test functions \( (p_x, q_x) \)
- 3-dimensional matrices of size \( (\Delta H \times P \times P) \)
- No on-the-fly quadratures!
- Flux computation on non-conformingly refined meshes reduces to evaluating matrix-vector products
- CPU time differs by a factor of several thousands for high orders and high refinement levels
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DG for Maxwell -- Adaptive mesh refinement

- **$h$-Refinement**
  - Approximation in interval $I_i$ is projected to intervals $I_{i,\text{I}}$ and $I_{i,\text{II}}$

- Local projection matrices $P_{\text{I}}$ and $P_{\text{II}}$:
  
  $P_{\text{I}}^{qp} = (\varphi^p, \psi^q_\text{I}) / (\psi^q_\text{I}, \psi^q_\text{I})$
  $P_{\text{II}}^{qp} = (\varphi^p, \psi^q_\text{II}) / (\psi^q_\text{I}, \psi^q_\text{II})$

\[
\begin{align*}
\mathbf{u}_{i,\text{I}} &= P_{\text{I}} \mathbf{u}_i \\
\mathbf{u}_{i,\text{II}} &= P_{\text{II}} \mathbf{u}_i
\end{align*}
\]
DG for Maxwell -- Adaptive mesh refinement

- **$h$-Coarsening**
  - The approximation in $I_i$ is considered piece-wise defined in $I_{i, I}$ and $I_{i, II}$

- **Definition of the projection matrix $P_c$**:
  
  \[
  P_{c, q} = \frac{\psi^q_I + \psi^q_{II}, \varphi^p}{(\varphi^p, \varphi^p)} = \frac{\psi^q_I, \varphi^p + (\psi^q_{II}, \varphi^p)}{(\varphi^p, \varphi^p)} = P_{c, q}^I + P_{c, q}^{II}
  \]

  \[
  u_i = P_{c, I} u_{i, I} + P_{c, II} u_{i, II}
  \]
DG for Maxwell -- Adaptive mesh refinement

- $p$-Adaptation
  - Mathematically trivial for sets of hierarchical basis functions
  - $p$-enrichment
    - add higher order coefficients
    - initialized by zero
  - $p$-reduction
    - delete high order coefficients
    - other coefficients remain unaltered
  - Memory and time efficient implementation ‘tricky’ though

- Stability of the full hp-adaptive algorithm in an energy norm proven [J Comp Appl Math, 2011]
DG for Maxwell -- Adaptive mesh refinement

- $\rho$-adaptive Simulation of a horn antenna

Contour plot of electric field in the horn
DG for Maxwell -- Adaptive mesh refinement

- $p$-adaptive Simulation of a horn antenna

Element orders for time instance of previous plot:
$P = 1, P = 2, P = 3, P = 4$
DG for Maxwell -- Adaptive mesh refinement

- \( p \)-adaptive Simulation of a horn antenna

<table>
<thead>
<tr>
<th>Method</th>
<th>Grid</th>
<th>( P_{\text{min}}/P_{\text{max}} )</th>
<th>Memory / MB</th>
<th>rel. Error</th>
<th>Comp. Time</th>
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<tr>
<td>DG</td>
<td>61 × 41 × 141</td>
<td>4 / 4</td>
<td>2007</td>
<td>—</td>
<td>328 min</td>
</tr>
<tr>
<td>FIT</td>
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<td>170</td>
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<td>31-72</td>
<td>0.039</td>
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</tbody>
</table>
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Example -- A horn antenna

A setup for testing dynamic mesh refinement

- **Step 1:** Find an initial mesh s.t. an error tolerance is met (< 1e-5)
Example -- A horn antenna

INFO: Grid information: #DoF (e+h): 57'276

Refinement strategy is **HP_ANISO: Anisotropic** $h$ and $p$ refinement
Example -- A horn antenna

INFO: Grid information: #DoF (e+h): 645'882

Refinement strategy is HP_ISO: Isotropic $h$ and $p$ refinement
Example -- A horn antenna

A setup for testing dynamic mesh refinement

- **Step 2**: Time-domain simulation

Broadband pulse (10-30 GHz)

Horn antenna (cut view)

Radar reflector
Example -- A horn antenna

A setup for testing dynamic mesh refinement

- Implementation of an error estimator based on reference solutions (as for obtaining the initial mesh) is underway
- Currently, solution based physical indicators are employed for controlling the refinement
- In this example it is energy density
Total propagation distance $\approx 60$ wavelengths

Up to 39 M DoF
(7.5 bn w/o adaptivity)
Summary

- Presented an efficient framework for performing $hp$-adaptive simulations with the DG method
- ... the underlying idea of $hp$-adaptivity
- ... but no details of the error or smoothness estimation (which is based on the concept of ‘reference solutions’) please come to speak with me if you are interested

- An $hp$-adaptive code should allow for anisotropic refinement as it yields substantial savings in computational resources
- A careful implementation has to be done in order to obtain reasonable performance
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