A discontinuous Galerkin method for studying elasticity and variable viscosity Stokes problems

Sascha M Schnepp, Dominic Charrier, Dave May
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Outline

- Model problem
- Continuous variational formulation
- Interior penalty discontinuous Galerkin methods
- Numerical tests
  - Convergence orders
  - Iteration counts for varying element aspect ratios and viscosity contrasts
- Summary and Outlook
Issues in Geodynamical Modelling

Regional scale

Outcrop scale

• Many length scales of interest
  \( O(1 \text{ m}) \rightarrow O(100 \text{ km}) \)

• Complex constitutive behaviour (ductile and brittle deformation)

• Large deformation

• Rheologically distinct layering (coefficient jumps)

Schmid et al. 2004

http://saturniancosmology.org/files/geology/sect2_1a.html
Governing equations

- Geodynamical processes are typically modelled as a **nonlinear incompressible Stokes problem**

\[-\nabla \cdot \left( \eta(u, p, t)(\nabla u + \nabla u^T) \right) + \nabla p = f(t)\]

\[\nabla \cdot u = 0\]

\[\frac{\partial}{\partial t} \eta + u \cdot \nabla \eta = 0\]

\[\frac{\partial}{\partial t} f_i + u \cdot \nabla f_i = 0\]

- After linearization and time-discretization, we solve a **linear** incompressible Stokes problem in each time step
Why use DG methods (for Stokes flow)?

- In contrast to FEM: DG uses element-wise continuous but **globally discontinuous approximations** of velocity and pressure (and any other quantity)
- Elements are coupled only by means of inter-element **numerical fluxes** across boundaries of **neighboring** cells
- Lifts constraints and gives DG methods their **high flexibility**
  - Work with basically any mesh – structured, unstructured, hybrid...
  - Local basis and degree can be chosen independently for each cell
  - Local $h$-, $p$- and $hp$-adaptivity is comparably easy to handle

**Inf-sup stable for**
- all $P_r - P_s$ and $Q_r - Q_s$ spaces with $s \leq r$
- high-order approximations on non-uniform and highly skewed meshes
Interior Penalty DG methods

Bilinear forms

\[
\begin{align*}
-\text{div} \ (\eta \ (\nabla u + \nabla u^T)) + \nabla p &= f & \text{in } \Omega \\
\text{div} \ u &= 0 & \text{in } \Omega
\end{align*}
\]

Galerkin procedure of cell-wise testing + summing over all cells

\[
\begin{align*}
a_h(\tilde{u}, \tilde{v}) + b_h(\tilde{v}, \tilde{p}) &= l_h(\tilde{v}) & \forall \tilde{v} \in V_h \\
b_h(\tilde{u}, \tilde{q}) &= 0 & \forall \tilde{q} \in Q_h
\end{align*}
\]

We obtain the following naive approach:

\[
a_h(\tilde{u}, \tilde{v}) = \sum_E \int_E \tau(\tilde{u}) : \varepsilon(\tilde{v}) \, dx - \sum_E \int_e \{\tau(\tilde{u}) n_e\} [\tilde{v}] \, ds \quad \times \text{not coercive}
\]

\[
b_h(\tilde{u}, \tilde{v}) = \sum_E \int_E -\text{div} \ \tilde{u} \ \tilde{q} \, dx + \sum_E \int_e \{q\} [\tilde{v} \cdot n_e] \, ds \quad \checkmark \text{inf-sup stable}
\]


\[
\varepsilon(u) = \frac{1}{2} (\nabla u + \nabla u^T) \quad [x] \text{ means jump of } x \text{ across element boundary}
\]

\[
\tau(u) = 2 \eta \varepsilon(u) \quad \{x\} \text{ means average of } x \text{ across element boundary}
\]
Interior Penalty DG methods
Nonsymmetric interior penalty DG (NIPG)

- Restoring coercivity \((a(x, x) \geq C\|x\|^2 \forall x)^*\): Oden, Babuška, Baumann (1998)

\[
a_h(\tilde{u}, \tilde{v}) = \sum E \int_E \tau(\tilde{u}) : \varepsilon(\tilde{v}) \, dx - \sum_e \int_e \{\tau(\tilde{u}) n^e\} [\tilde{v}] \, ds + \sum_e \int_e \{\tau(\tilde{v}) n^e\} [\tilde{u}] \, ds
\]

\((✓)\) coercive

- Forcing (more) continuity of the DG solution:

\[
a_h(\tilde{u}, \tilde{v}) = \sum E \int_E \tau(\tilde{u}) : \varepsilon(\tilde{v}) \, dx - \sum_e \int_e \{\tau(\tilde{u}) n^e\} [\tilde{v}] \, ds + \sum_e \int_e \{\tau(\tilde{v}) n^e\} [\tilde{u}] \, ds
\]

\[+ \sum_e \alpha_e \int_e [\tilde{u}] [\tilde{v}] \, ds\]

term zero for continuous \(u\)

Rivière, Wheeler, Girault (1999)

* in the appropriate norm
Interior Penalty DG methods
Symmetric interior penalty DG (SIPG)

- Changing one sign in NIPG form...

\[
a_h(\tilde{u}, \tilde{v}) = \sum_E \int_E \tau(\tilde{u}) : \varepsilon(\tilde{v}) \, dx - \sum_e \int_{e} \{ \tau(\tilde{u}) n^e \} [\tilde{v}] \, ds
\]

\[
- \sum_e \int_{e} \{ \tau(\tilde{v}) n^e \} [\tilde{u}] \, ds
\]

\[
+ \sum_e \alpha_e \int_{e} [\tilde{u}] [\tilde{v}] \, ds
\]

\checkmark \text{coercive and symmetric for } \alpha_e > \alpha_{e,\text{min}} > 0
Test case SolCx: Large viscosity jump

- Flow driven by temperature field
  \[ f = (0, \sin(n_y \pi y) \cos(\pi x))^T \]
- Free-slip boundary conditions
- Analytic solution (SolCx) approximated by truncated series

\[ \eta_a : \eta_b = 1 : 10^6, \ x_c = 0.5, \ n_y = 3 \]

May, Moresi (2008)
Test case SolCx: Large viscosity jump
Illustration of a low order solution

\[ Q_1 - Q_0 \]

no spurious features

velocity

pressure
**Test case SolCx: Large viscosity jump**

Velocity error ($L^2$-norm)

- **SIPG**
  - Optimal conv. in $L^2$-norm
  
- **NIPG**
  - Optimal conv. in $L^2$-norm for odd $k$
  - Suboptimal conv. for even $k$

\[
\| u - u_h \|_{L^2(\Omega)}
\]

- $Q_1 \rightarrow Q_0$, $Q_2 \rightarrow Q_1$
- $Q_3 \rightarrow Q_2$, $Q_5 \rightarrow Q_4$

![Graphs showing velocity error for SIPG and NIPG](image)

- Parameter values: $1.96$, $3.02$, $3.97$, $5.71$, $2.02$, $2.07$, $3.97$, $6.12$
Test case SolCx: Large viscosity jump
Pressure error ($L^2$-norm)

- Optimal conv. in $L^2$-norm

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|p - p_h|_{L^2(\Omega)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>$10^{-2}$</td>
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<tr>
<td>2.00</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>3.00</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>4.96*</td>
<td>$10^{-8}$</td>
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</table>

**SIPG**

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|p - p_h|_{L^2(\Omega)}$</th>
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</thead>
<tbody>
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<td>1.13</td>
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<tr>
<td>3.16</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>5.16*</td>
<td>$10^{-8}$</td>
</tr>
</tbody>
</table>

**NIPG**

$Q_1$ — $Q_0$, $Q_2$ — $Q_1$, $Q_3$ — $Q_2$, $Q_5$ — $Q_4$
NIPG is a stable discretization for Stokes flow

- **Penalty parameters trivial**: \( \alpha_e > 0 \)
- Optimal pressure convergence
- Local mass conservation
- In general, **no optimal \( L^2 \)-convergence of the velocity**
- Resulting matrix is **non-symmetric**: Solvers for symmetric matrices (CG,...) are not applicable

SIPG is a stable discretization for Stokes flow for sufficiently large penalty parameters

- Optimal pressure convergence
- **Optimal \( L^2 \)-convergence of the velocity**
- Local mass conservation
- Resulting matrix is **symmetric**: Solvers for symmetric matrices (CG,...) are applicable

- **Penalty parameter choice is non-trivial** but lower bound can be estimated
Test case SolCx: Large viscosity jump

Iteration counts for varying aspect ratio

- Testing dependency of inf-sup constant from element aspect ratio through number of iterations in a solve
- SolCx test as before (viscosity contrast 1:10^6)
- Solver setup I:
  - outer solver fgmres
  - preconditioner:
    - Approximate LDU factorization
  - velocity solver:
    - direct solve (LU)
  - pressure Schur complement solver:
    - GMRES / Jacobi
- **Very modest increase** of number of iterations

<table>
<thead>
<tr>
<th>Aspect ratio</th>
<th>Outer solver iterations</th>
<th>Schur complement iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
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<td>256</td>
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Test case SolCx: Large viscosity jump

Iteration counts for varying aspect ratio

- Testing dependency of inf-sup constant from element aspect ratio through number of iterations in a solve

- SolCx test as before (viscosity contrast 1:10^6)

- Solver setup II:
  - outer solver fgmres
  - preconditioner (Elman, Silvester, Wathen):
    - Approximate DU factorization
  - velocity solver:
    - direct solve (LU)
  - pressure Schur complement solver:
    - only Jacobi preconditioner

- Very modest increase of number of iterations

<table>
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<th>Schur complement iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
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</tr>
<tr>
<td>256</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>
Test case SolCx: Large viscosity jump
Iteration counts for varying viscosity contrasts

- Testing dependency of inf-sup constant from element aspect ratio through number of iterations in a solve

- SolCx test BUT Variation of the viscosity contrast
  - Solver setup I
  - 64 x 64 mesh (insensitive)
  - outer solver (rel) tolerance: 1e-5
  - pressure complement solver (rel) tolerance: 1e-10

- Very modest increase of number of iterations

<table>
<thead>
<tr>
<th>Viscosity contrast</th>
<th>Outer solver iterations</th>
<th>Schur complement iterations (avg)</th>
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<tbody>
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<td>1</td>
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<tr>
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<td>1</td>
<td>5</td>
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<tr>
<td>$10^4$</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>$10^6$</td>
<td>1</td>
<td>6</td>
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<tr>
<td>$10^8$</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>$10^{10}$</td>
<td>2</td>
<td>11</td>
</tr>
</tbody>
</table>
Summary and Outlook

- We implemented **two high-order DG methods** for variable viscosity Stokes (NIPG and SIPG) (deal.ii testbed and PETSc implementation as a community tool to be released open-source)
- The methods **show the desired properties**
  - inf-sup stability
  - high accuracy
  - robustness w.r.t. parameter jumps and element aspect ratios
- We observed **no spurious pressure modes** in our tests
- IPDG methods are suitable for local anisotropic $h$- and $p$-refinement
  - We expect that this will help us dealing with discontinuous material coefficients
- Extension towards time-dependent problems underway
- Multi-level preconditioner to facilitate 3D calculations
- We hope to publish the detailed method in summer
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